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Hermite–Padé Approximants to Functions Meromorphic on a Riemann Surface

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The asymptotic form of Hermite–Padé approximants to a set of m functions each meromorphic on a Riemann surface with m sheets is determined.

1. INTRODUCTION

Our understanding of Padé approximants to functions with branch points has been greatly aided by the existence of more or less explicit constructions of the approximants for certain functions. Dumas [1], in his unfortunately mostly neglected thesis, gave a careful study for certain functions involving the square root of a quartic polynomial. The explicit calculation of polynomials orthogonal with respect to a weight of the form $(t^2-1)^{-1/2}\times$ (inverse of a polynomial) forms the basis of Szegö's [2] investigation of the asymptotic form of such polynomials, which leads immediately to results on the convergence of Padé approximants to the corresponding functions. Akhiezer [3] generalized the work of Dumas and Szegö to the calculation of polynomials orthogonal with respect to $X(t)^{-1/2}\times$ (inverse of a polynomial). The same device was used by Nuttall and Singh [4], who were unaware of Akhiezer's work, in their study of the convergence of Padé approximants to certain functions cut along a set of minimum capacity.

Recently interest has been growing in a generalization of Padé approximants that we shall call Hermite–Padé approximants [5]. They have been used in areas ranging from the theory of phase transitions [6] to number theory [7]. Chudnovsky [7] has studied their relation to the inverse monodromy problem and has constructed explicit forms in certain cases. The study of Hermite–Padé approximants, under a variety of names, goes back to Hermite [8] and Padé [9]. Some more recent work has been performed by Mahler [10] and de Bruin [11].

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In this paper we show how to generalize the methods described in the first paragraph to give fairly explicit information on Hermite–Padé approximants to certain functions. The dominant part of the asymptotic behavior of the Hermite–Padé polynomials is given. Our work leads us to suggest that the asymptotic behavior for a larger class of functions may be related to the solution of a generalized Hilbert problem.

2. Construction of Hermite–Padé Approximants

We shall be concerned throughout with functions meromorphic on a Riemann surface $\mathscr{R}$ corresponding to an equation $R(y, z) = 0$, where $R(y, z)$ is an irreducible polynomial in y, z of degree m in y. (See [12] for a full account of the necessary theory of Riemann surfaces.) The Riemann surface may be thought of as consisting of m copies of the complex z-plane joined together across various cuts running between branch points. A branch point is a point on $\mathscr{R}$ where $\partial R/\partial y = 0$. The point at ∞ may be a branch point, but we shall assume that this is not so on the first sheet. The notation $z^{(i)}$ will mean the point z on sheet i.

There is at each point on $\mathscr{R}$ a local variable, which is z away from ∞ and branch points. Near ∞ it is z^{-1} and at a branch point a it is $(z-a)^{\nu}$, $\nu = r^{-1}$, if r sheets meet at a. A function meromorphic on $\mathscr{R}$ is single-valued on $\mathscr{R}$ and meromorphic in the local variable at each point. Each Riemann surface has a genus $g \geqslant 0$. The theory shows that any meromorphic function has the same number of zeros as poles. Suppose that we choose the location of the poles of a meromorphic function. Then we may choose the location of all but g of the zeros. The remaining g zeros have to satisfy the condition of Abel's theorem.

Now given functions $F_i(x)$, $i = 1,..., m$, (note m is the number of sheets in $\mathscr{R}$) analytic in a neighborhood of $x = 0$, we construct polynomials $P_i(x)$ of degree μ_i, $i = 1,..., m$ according to

$$\sum_{i=1}^{m} P_i(x)\, F_i(x) = O(x^{\mu_1 + \cdots + \mu_m + m - 1}). \tag{1}$$

The polynomials P_i are the Hermite–Padé polynomials and always exist, although they may not be unique. We shall restrict our attention to the case $\mu_i = \mu$, $i = 1,..., m$. To achieve a slight simplification and to assist comparison with the formula for vector orthogonal polynomials [5] we shall rewrite (1) as

$$\sum_{i=1}^{m} p_i(z)\, f_i(z) = O(z^{-(m-1)\mu - m + 1}), \tag{2}$$

where $p_i(z) = z^\mu P_i(z^{-1})$, $f_i(z) = F_i(z^{-1})$. In addition we take $f_1 = 1$ with no loss of generality for our purposes.

Suppose now that $f_i(z)$ are meromorphic on $\mathscr{R}$ and that $f_i(z)$ in (2) refer to $f_i(z^{(1)})$. Assume also that their poles are contained in the set $a_j, j = 1,... \lambda$, where a_j are points on $\mathscr{R}$, repeated in case an f_i has a higher order pole. Of course no $f_i(z)$ has a pole at $\infty^{(1)}$. To avoid unessential complications we assume that $\infty^{(j)}$, $j = 1,..., m$ are not branch points of $\mathscr{R}$.

For a particular solution of (2) we consider the function

$$Q(z) = \sum_{i=1}^{m} p_i(z) f_i(z).$$

It is meromorphic on $\mathscr{R}$ and has a zero of order $(m-1)\mu + m - 1$ at $\infty^{(1)}$. In addition it may have poles at $\infty^{(j)}$, $j = 2,..., m$ of order μ and at a_j, $j = 1,..., \lambda$. Thus it has at most $(\lambda - (m-1))$ additional zeros which occur at b_j, $j = 1,..., \lambda - m + 1$, say. Once the b_j are known Q is determined up to normalization. We set $c_j = b_j$, $i = 1,..., \lambda - m + 1$ and $c_j = \infty^{(1)}$, $j = \lambda - m + 2,..., \lambda$.

Given $Q(z)$, the polynomials $p_i(z)$ may be found by solving

$$\sum_{i=1}^{m} f_i(z^{(j)})\, p_i(z) = Q(z^{(j)}), \qquad j = 1,..., m. \tag{3}$$

If we define the functions $H_{ij}(z)$ as the solution of

$$\sum_{i=1}^{m} f_i(z^{(k)})\, H_{ij}(z) = \delta_{kj}, \qquad j, k = 1,..., m \tag{4}$$

then we have

$$p_i(z) = \sum_{j=1}^{m} H_{ij}(z)\, Q(z^{(j)}), \qquad i = 1,..., m. \tag{5}$$

To make sense of our discussion we shall assume that

$$\det[f_i(z^{(j)})] \neq 0. \tag{6}$$

In Section 4 an interesting observation will be based on the following lemma.

LEMMA. *We may write $H_{ij}(z) = H_i(z^{(j)})$, where $H_i(z)$ is meromorphic on $\mathscr{R}$.*

Proof. Since $f_i(z)$ is meromorphic we may write [12]

$$f_i(z) = \sum_{l=1}^{m} T_{li}(z)\, y^{l-1}, \tag{7}$$

where $T_{ij}(z)$ is a rational function of z. Thus (4) becomes, writing $y_k = y(z^{(k)})$,

$$\sum_{l=1}^{m} y_k^{l-1} \sum_{i=1}^{m} T_{li}(z)\, H_{ij}(z) = \delta_{kj}, \qquad j, k = 1,..., m,$$

or in matrix form

$$VTH = I,$$

where the Vandermonde matrix V is given by

$$V_{ij} = y_i^{j-1}.$$

We deduce that

$$V^T VTH = V^T. \tag{8}$$

Now

$$(V^T V)_{ij} = \sum_{k=1}^{m} y_k^{i+j-2},$$

which is a rational function of z. Moreover det $V = 0$ only at branch points, so that the inverse of $V^T V$ is a matrix with elements that are rational in z, having poles only at the branch points. Also, from (7)

$$\det[f_i(z^{(j)})] = \det T \det V,$$

which means from (6) that det $T \not\equiv 0$. Thus $W = (V^T VT)^{-1}$ is a matrix with elements that are rational in z, having poles only at the branch points and the zeros of $\det[f_i(z^{(j)})]$.

The solution of (4) is

$$H_{ij}(z) = \sum_{k=1}^{m} W_{ik}\, y_j^{k-1},$$

which exhibits $H_{ij}(z)$ as the value on sheet j of a meromorphic function, and the lemma follows.

3. Asymptotic Form of $p_i(z)$

For any two given distinct points z_1, z_2 on $\mathscr{R}$ there exists a unique differential $dE(z_1, z_2)$ of the third kind whose only singularities are simple poles at z_1, z_2 with residues 1, -1, respectively, and such that the periods of the integral $E(z_1, z_2; z)$ are pure imaginary [12]. The function

$$\ln Q(z) - \mu \sum_{j=2}^{m} E(\infty^{(1)}, \infty^{(j)}; z) - \sum_{j=1}^{\lambda} E(c_j, a_j; z)$$

will have no singularities and periods which are pure imaginary (since $Q(z)$ is single-valued on $\mathscr{R}$) and therefore must be constant. It follows that

$$Q(z) = \exp(\mu\phi(z))\, \psi(z),$$

where

$$\phi(z) = \sum_{j=2}^{m} E(\infty^{(1)}, \infty^{(j)}; z)$$

and

$$\psi(z) = \exp\left(\sum_{j=1}^{\lambda} E(c_j, a_j; z)\right).$$

We see that $\phi(z)$ is the Abelian integral of the third kind, unique up to an additive constant, having pure imaginary periods and having simple poles at $\infty^{(j)}$, $j = 1,..., m$, where the residues are $(m-1)$, $j = 1$, and -1, $j = 2,..., m$.

With these preliminaries we are in a position to prove the following theorem.

Theorem. *Suppose that we are given B, a simply connected closed bounded domain in the complex plane, not containing any branch points of $\mathscr{R}$, with boundary consisting of a finite number of analytic arcs, and an integer k, $1 \leqslant k \leqslant m$, such that*

$$\operatorname{Re}\phi(z^{(k)}) > \operatorname{Re}\phi(z^{(j)}), \qquad z \in B, j \neq k.$$

Then

$$Q(z^{(j)})/Q(z^{(k)}) \to 0, j \neq k$$

and

$$p_i(z)/p_j(z) \to H_{ik}/H_{jk}$$

the convergence being in capacity, $z \in B$, as $\mu \to \infty$. (The notion of convergence in capacity was first used in connection with Padé approximants by Pommerenke [13].*)*

Proof. The proof is similar to that of Lemma 7.3 of [4]. Let us define $h(z)$ as one value of

$$h(z)=\psi(z^{(k)})/\psi(z^{(j)})$$

so that $h(z)$ is single-valued meromorphic in B with at most λ poles and zeros. The value of $|h(z)|$ is the same no matter which determination was chosen.

Define $\delta(\eta)>0$ for sufficiently small $\eta>0$ by

$$\operatorname{Cap}\{z:|h(z)|\leqslant\delta(\eta)\}=\eta.$$

Now the only dependence of $h(z)$ on μ is through the location of the points c_j, $a_j\in\mathscr{R}$. For a given η it must happen that either there exists $\delta_0(\eta)>0$ such that $\delta(\eta)>\delta_0(\eta)$ for all μ or that there is a sequence of values of $\mu, \mu_1<\cdots<\mu_l\cdots$ such that $\delta(\eta)\to 0$ as $l\to\infty$. We show that the latter is impossible, for in this case there would be a subsequence for which each c_j, a_j approached limits (in the metric provided by the local variable on $\mathscr{R}$). There would be a corresponding limit function $h_0(z)$, meromorphic in B satisfying

$$\operatorname{Cap}\{z:|h_0(z)|\leqslant 0\}=\eta$$

from which it follows that $h_0(z)=0$, a contradiction.

We deduce that for $z\in B$

$$|Q(z^{(j)})/Q(z^{(k)})|<\exp[\mu(\operatorname{Re}\phi(z^{(j)})-\operatorname{Re}\phi(z^{(k)}))]\,\delta_0^{-1}(\eta)$$

except for a set of capacity η, and the theorem follows immediately.

We remark that, for large μ, almost all the zeros of $p_i(z)$ will approach the curve $S=\{z\in\mathbb{C}:\operatorname{Re}\phi(z^{(k)})=\operatorname{Re}\phi(z^{(j)});\ \operatorname{Re}\phi(z^{(l)})\leqslant\operatorname{Re}\phi(z^{(k)})$ for all $l\neq k, j\}$, since assumption (6) leads to the conclusion that neither of H_{ik}, H_{ij} can be identically zero.

4. Relation to Generalized Hilbert Problem

If we define the meromorphic function $\chi_i(z)$ as

$$\chi_i(z)=H_i(z)\,Q(z),\qquad z\in\mathscr{R},$$

we see from (5) that

$$p_i(z)=\sum_{j=1}^{m}\chi_i(z^{(j)}),\qquad i=1,\ldots,m.$$

Let us denote by $k \neq 1$ that sheet for which $\operatorname{Re}\phi(z^{(j)})$ is largest, i.e.,

$$\operatorname{Re}\phi(z^{(k)}) > \operatorname{Re}\phi(z^{(j)}), \qquad j \neq k.$$

Thus the boundary of sheet k is S.

A generalization of Hilbert's problem can be set up on $\mathscr{R}$ in the following form. Solve for $\chi_i(z)$, $i = 1,..., m$, meromorphic on sheet k, and for $Q(z)$, meromorphic on all of $\mathscr{R}$ except sheet k, with appropriate behavior at ∞, the system

$$\sum_{i=1}^{m} f_i(z^{(j)})\,\chi_i(z) = 0, \qquad j \neq k, \tag{9}$$

$$\sum_{i=1}^{m} f_i(z)\,\chi_i(z) = Q(z), \qquad z \in S. \tag{10}$$

It follows from (4) that the $\chi_i(z)\,(=\chi_i(z^{(k)}))$, $Q(z)$ of Section 2 indeed obey (9) and (10).

We shall not attempt to determine what, if any, further conditions need be imposed in order to make the solution of the problem unique. Our purpose is to suggest that the solution of such a problem could well give the correct asymptotic form of $p_i(z)$ $(\sim\chi_i(z))$ for situations more general than that in which $f_i(z)$ is meromorphic on $\mathscr{R}$. For a number of cases when $m = 2$ (ordinary Padé approximants) we know that the suggestion is correct, as is also the case for some examples with $m > 2$ studied by Chudnovsky [7].

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